



TITLE:

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AUTHOR(S):

Tsurumi, Kazuyuki; Sekine, Tadayuki

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Integral means of holomorphic mappings in C^n

Kazuyuki Tsurumi
(東京電機大工 鶴見和之)

Tadayuki Sekine
(日本大学薬 関根忠行)

Let $f(z)$ and $g(z)$ be holomorphic functions in the unit disk U with $f(0) = g(0) = 0$.

The function $g(z)$ is said to be subordinate to the function $f(z)$ if there exists a function $\psi(z)$ holomorphic in U such that $|\psi(z)| \leq |z|$ for $z \in U$ and $g(z) = f(\psi(z))$. Let $g(z)$ be a holomorphic function in U . For $0 < p < \infty$ and $0 \leq r < 1$, let us put

$$M_p(r, \varphi) := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}.$$

Then we have the theorem

Theorem A (The subordination theorem of Littlewood [2], p.191)

Suppose that f and g are holomorphic in U and $f(0) = g(0) = 0$ and that g is subordinate to f , then we have

$$M_p(r, g) \leq M_p(r, f) \quad (0 < p < \infty, \quad 0 \leq r < 1)$$

The purpose of this note is to extend this theorem to the case of C^n .

§ 1. Preliminaries

Let us denote a point z of the space C^n by the column vector

$$z := \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

and z^* denotes the conjugate transposed vector of z . The norm of z is denoted by

$\|z\| := \sqrt{z^* z}$. Denoted by $B_n(r, z_0)$ the ball in C^n with radius r and center z_0 , i.e.,

$$B_n(r, z_0) := \{z \in C^n \mid \|z - z_0\| < r\},$$

and let $B := B_n(1, 0)$, $S(r) := \partial B_n(r, 0)$ (the boundary of $B_n(r, 0)$) and $S := S(1)$.

For $z \in C^n$, we will use the following differential forms and operators for the proof of our theorem;

$$dz = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix}, \quad d\bar{z} = \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix}.$$

$$\omega(z) := dz_1 \wedge \cdots \wedge dz_n \quad (\text{n-times})$$

$$\eta(z) := \sum_{j=1}^n (-1)^{j+1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n \quad (\text{Leray form})$$

$$d\sigma(z) := \frac{1}{(2i)^n} \{ \eta(z) \wedge \omega(z) + (-1)^{q(n)} \omega(z) \wedge \eta(z) \}. \quad (q(n) = \frac{n(n-1)}{2}).$$

Let $z = r\zeta$, $r := \|z\|$ and $\|\zeta\| = 1$. Then we have

$$\bar{\zeta}_1 d\zeta_1 + \cdots + \bar{\zeta}_n d\zeta_n + \zeta_1 d\bar{\zeta}_1 + \cdots + \zeta_n d\bar{\zeta}_n = 0.$$

Thus $d\sigma(z) = r^{2n-1} d\sigma(\zeta)$, and the form $dS(\zeta) := \frac{1}{v(S)} d\sigma(\zeta)$ is the normalized

rotation invariant surface measure on S , where

$$v(S) = \frac{2\pi^n}{(n-1)!} \quad (\text{the area of } S).$$

$$dv(z) := \frac{1}{(2i)^n} \omega(z) \wedge \omega(z) \quad (\text{the volume element of } C^n).$$

Then we get

$$dv(z) := \frac{1}{(2i)^n} \{ \eta(\zeta) \wedge \omega(\zeta) + (-1)^{q(n)} \omega(\zeta) \wedge \eta(\zeta) \} \wedge r^{2n-1} dr$$

$$= d\sigma(\zeta) \wedge r^{2n-1} dr$$

Let us set

$$\frac{\partial}{\partial z} := \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right), \quad \frac{\partial}{\partial z^*} := \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}$$

$$\Delta := 4 \frac{\partial^2}{\partial z^* \partial z} = 4 \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

$$H := \frac{\partial^2}{\partial z^* \partial z} = \begin{pmatrix} \frac{\partial^2}{\partial \bar{z}_1 \partial z_1} & \dots & \frac{\partial^2}{\partial \bar{z}_1 \partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial \bar{z}_n \partial z_1} & \dots & \frac{\partial^2}{\partial \bar{z}_n \partial z_n} \end{pmatrix}$$

Let $u(z)$ be a real function in a domain D in \mathbb{C}^n . The function $u(z)$ is said to be subharmonic in D if the following three conditions hold :

- (1) $-\infty \leq u(z) < \infty$
- (2) $u(z)$ is upper semicontinuous in D
- (3) For any point $z_0 \in D$, we can take an $r > 0$ such that

$$B(r, z_0) \subset D \text{ and}$$

$$u(z_0) \leq \frac{1}{v(S)} \int_{S(r, z_0)} u(\zeta) d\sigma(\zeta)$$

Suppose that $u(z)$ is a function of the class C^2 in D . Then

$$u(z) \text{ is } \left\{ \begin{array}{l} \text{subharmonic} \\ \text{plurisubharmonic} \end{array} \right\}$$

$$\Downarrow \Uparrow$$

$$\left\{ \begin{array}{l} \Delta u = 4 \frac{\partial^2 u}{\partial z^* \partial z} \geq 0 \\ H(u) = \frac{\partial^2 u}{\partial z^* \partial z} \text{ (complex hessian) is positive definite.} \end{array} \right.$$

Thus the plurisubharmonic function is subharmonic.

The mapping function $f(z)$ from a domain in C^n to C^n is denoted by the column vector

$$f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}.$$

The mapping $f(z)$ is said to be holomorphic if each component functions $f_j(z)$ ($j = 1 \dots n$) are holomorphic. Let $H_{n,m}$ be the family of holomorphic mappings from B_n to C_m and suppose that $f(z)$ and $g(z)$ are belonging to $H_{n,m}$ and that $f(0) = g(0) = 0$.

The mapping $g(z)$ is said to be subordinate to $f(z)$ if there exists a holomorphic mapping $\Psi(z)$ from B_n to B_n such that $|\Psi(z)| \leq |z|$ ($z \in B_n$) and $g(z) = f(\Psi(z))$.

For a mapping $f(z) \in H_{n,m}$, we set

$$M_p(r, f) := \left\{ \frac{1}{v(S_r)} \int_{S_r} |f(z)|^p d\sigma(z) \right\}^{\frac{1}{p}} \quad (0 < r < 1, 0 < p < \infty).$$

§ 2. Theorems for subharmonic functions

For the ball $B_n(z_0, r) \subset C^n$, let us put

$$K(\xi, z) := \frac{1}{v(S)} \frac{r^2 - |z - z_0|^2}{r |z - \xi|^{2n}}$$

Then the following theorems hold :

Theorem B ([3], p.32, Theorem 1.16)

Let $\varphi(z)$ be a continuous function on $S(r, z_0)$. Let us put

$$u(z) := \int_{S(r, z_0)} K(\xi, z) \varphi(\xi) d\sigma(\xi) \quad (z \in B(r, z_0))$$

Then the function $u(z)$ is the solution of the problem of Dirichlet for $B(r, z_0)$ with the boundary value $\varphi(z)$.

Theorem C ([3], p.52, Theorem 2.7)

Let $\varphi(z)$ be a subharmonic function on a domain D in C_n and $u(z)$ is not equal to $-\infty$. Suppose that $B(z_0, r) \subset D$. Let us put

$$V(z) := \chi_{B(r, z_0)}(z) \int_{S(r, z_0)} K(\xi, z) \varphi(\xi) d\sigma(\xi) + \chi_{D - \overline{B(r, z_0)}}(z) u(z)$$

where χ_A denotes the characteristic function for A . Then $V(x)$ is subharmonic in D and harmonic in $B(r, z_0)$, and we have $u(x) \leq V(x)$ in $B(r, z_0)$.

Main Theorem. Let $\varphi(z)$ be a subharmonic function in B and let $\psi(z)$ be a holomorphic mapping from B to B such that $\|\Psi(z)\| \leq \|z\|$. Then we have

$$\int_{S_n(r, 0)} \varphi(\psi(z)) d\sigma(z) \leq \int_{S_n(r, 0)} \varphi(z) d\sigma(z)$$

As the corollary of our theorem, we obtain the Subordination Theorem of Littlewood for C^n .

Corollary 1. Let f and g be holomorphic mappings of $H_{n,m}$ such that $f(0) = g(0) = 0$. Suppose that g is subordinate to f , then we have

$$M_p(r, g) \leq M_p(r, f) \quad (0 < p < \infty, 0 \leq r < 1)$$

From Corollary 1, we get the following :

Corollary 2. Let the functions $f(z)$ and $g(z)$ be the same as in Corollary 1, then we have

$$\int_{B_n} \|g(z)\|^p dv(z) \leq K \int_{B_n} \|f(z)\|^p dv(z)$$

where K is a constant.

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